# Discrete Time Representation of Non-stationary Continuous Time Models with Unequally Spaced Data

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**Abstract**

This paper presents an exact discrete time representation of non-stationary continuous time systems with unequally spaced flows or a mixture of stocks and flows. The approach to obtain the exact discrete time representation with flow variables does not depend on the continuous time parameter matrix being non-singular, namely the underlying continuous time system may be non-stationary. In both cases the exact discrete time representations follow a VARMA(1, 1) process with time-varying parameters and heteroskedasticity, despite that the underlying continuous time model has constant parameters and homoskedasticity. The time-varying parameters and the heteroskedastic variance arise due to the variations in the sampling intervals, whereas the moving average disturbances arise due to the flow nature of the observations. A Monte Carlo simulation on estimation of a cointegrated continuous time system with unequally spaced flows is conducted, aiming at assessing estimate properties when unequal sampling intervals are correctly accounted for. Simulation evidence indicates the favour of exact discrete time models accounting for the irregularity of sampling intervals.

# Contents

# 1 INTRODUCTION

Estimating continuous time models based on the exact discrete time analogue has been a popular topic in time series analysis for decades. Most research in estimations of continuous time models assume data are observed over the same interval, which is often time normalised as unity. The fact that some data are not observed on a regular basis has drawn some attention, for instance, Robinson (1977) pointed out the possibility for modelling irregularly sampled time series.

Unequally spaced data can be found in a number of fields including economics and finance. A leading example can be found in monthly data, in which observation intervals may vary with the variation in the length of calendar months ranging from 28 days to 31 days with roughly 10 percent difference. Unequally spaced data could also appear in financial data, such as data on trades that take place infrequently. In addition, for daily closing price of stock exchange, weekends and public holidays would lead to the irregularity in the sampling intervals. Such type of data could also be obtained in other fields such as the timing of elections, which does not happen on regular basis, in political science.

Some research has addressed the issue in estimating continuous time models with unequally spaced data. One approach to estimate such models would be adopting state space representations. For example, in Harvey and Stock's research (1985) they estimate continuous time autoregressive systems using Kalman filter recursions. Their study is further extended to allow for exogenous variables and mixed frequency in (unequally spaced) data by Zadrozny (1988). Harvey and Stock (1993) later provide estimation of continuous time structural time series model where data are stocks, flows or a mixture of both that are unequally spaced. In Koopman et al (2018), they estimate continuous time structural models via the state space approach with high frequency traffic data observed at unequally spaced points in time.

In previous work I provide the derivation of exact discrete time representations of continuous time systems when data are unequally spaced. Exact discrete time representations are provided in three cases: when data are purely stock variables, purely flow variables, or mixed of both stocks and flows. In all cases the exact discrete time representations exhibit time-varying parameters and heteroskedasticity. When data are purely stock variables or a mixture of stocks and flows, the exact discrete time representations require the underlying continuous time system to be stationary. Such restriction would limit the applications to non-stationary systems such as unit root or cointegrated systems.

The focus of this chapter is on providing an approach to derive exact discrete time representation of non-stationary continuous time systems with unequally spaced flows and mixed data. The approach does not impose restrictions on the continuous time parameter matrix. The discrete time representation is exact and is applicable to non-stationary systems as well. Despite that the underlying continuous time system has constant parameters and is homoskedastic, the exact discrete time representations, in both cases, follow a VARMA(1,1) process with time-varying parameters and heteroskedasticity. Such a scenario arises when the continuous time system is observed at unequally spaced intervals. Both time-varying parameters and heteroskedastic variances arise due to the variations in the sampling intervals, whereas the moving average disturbances arise due to the flow nature of the observations.

In the following, section 2 provides derivation of the exact discrete time representation of a continuous time system where the variables are observed over unequally spaced discrete intervals. The model considered is multivariate and includes a deterministic time trend. The discrete time representation has time-varying parameters and heteroskedsticity. In particular, the disturbance vector is a time-varying moving average, in which the covariance matrix is time dependent.

Section 3 considers the case where the variables of interest are a mixture of stocks and flows, in which case the discrete time representation relies on the assumption that a sub-matrix of the continuous time parameter is non-singular (hence is invertible). This assumption, although limiting the potential applications, for example, to systems involving zero roots, is weaker than many that have appeared in the literature to date. The discrete time representation also has time-varying parameters and heteroskedastic moving average disturbances.

Results of a Monte Carlo simulation study are reported in Section 4. The study considers a cointegrated system of flow variables whose sampling intervals coincide with the variation of calendar months. Simulation results indicate that estimation bias is reduced when the unequal sampling intervals are correctly accounted for (rather than assuming all intervals are the same). Section 5 contains some concluding comments and detailed Monte Carlo simulation procedures are provided in the Appendix.

## 2 AN EXACT DISCRETE TIME MODEL WITH FLOWS

This section provides derivations of discrete time representation of a continuous model. [1] The continuous time model is a system of first-order stochastic differential equations with flow variables and stochastic trends.

Let $x(t)$ be an $n \times 1$ stochastic process generated by

$$dx(t) = [\mu + \gamma t + Ax(t)]dt + \zeta(dt), \quad t > 0, \tag{1}$$

where $\mu$ and $\gamma$ are $n \times 1$ parameter vectors, $A$ is an $n \times n$ matrix, and $\zeta(dt)$ is an $n \times 1$ vector of random measures satisfying:

**Assumption 1**.

$$E[\zeta(dt)] = 0$$

$$E[\zeta(dt)\zeta(dt)'] = \Sigma dt,$$

where $\Sigma$ is an unknown symmetric positive definite matrix such that $\Sigma = \sigma^2 I_n$ (with $\sigma^2$ being some random variable) and

$$E[\zeta_i(\Delta_1)\zeta_j(\Delta_2)'] = 0,$$

for $i, j = 1, 2, \cdots, n$; $i \neq j$; and $\Delta_1 \cap \Delta_2 = \emptyset$.

In what follows, it is assumed that samples are observed at the points $t_i$ $(i = 1, \ldots, T)$ such that $0 < t_1 < \ldots < t_T$ and $t_i = t_{i-1} + \delta_i$ for some $\delta_i > 0$ $(i = 1, \ldots, T)$. In the case of a stock variable the sequence of observations is of the form

$$x(t_1),\ x(t_2),\ \cdots,\ x(t_N). \tag{2}$$

Extensive use is made of the matrix exponential and various functions thereof. The matrix exponential is defined as

$$e^A = \sum_{j=0}^{\infty} \frac{1}{j!} A^j,$$

[1] The method for deriving the exact discrete time model with flows follows the joint paper with my supervisor-Time-Varying Parameters and Heteroskedasticity: Continuous Time Systems with Unequally-Spaced Data.

and it is convenient to define the matrix functions

$$
\begin{aligned}
F(z) &= e^{Az}, \\
G(z) &= \int_0^z e^{As} ds, \\
H(z) &= \int_0^z s e^{As} ds, \\
J(z) = \int_0^z G(s) ds &= \int_0^z \left( \int_0^s e^{Ar} dr \right) ds, \\
K(z) = \int_0^z H(s) ds &= \int_0^z \left( \int_0^s r e^{Ar} dr \right) ds, \\
M(z) = \int_0^z s G(s) ds &= \int_0^z s \left( \int_0^s e^{Ar} dr \right) ds,
\end{aligned}
$$

in all cases $z$ is a known constant. In particular, when $z = \delta_i$ the particular matrices are defined as

$$F_i = F(\delta_i), G_i = G(\delta_i), H_i = H(\delta_i), J_i = J(\delta_i), K_i = K(\delta_i), M_i = M(\delta_i), (i = 1, \ldots, T).$$

In the case of unequally spaced stock variables (when $x(t)$ is a stock variable), based on results from Theorem 2.1 in my PhD thesis [2], the discrete time representation of (1) is obtained as

$$x(t_i) = c_{0i} + c_{1i} t_i + F_i x(t_{i-1}) + \eta(t_i), \quad i = 1, \ldots, T, \tag{3}$$

where $c_{0i} = G_i \mu - H_i \gamma, \quad c_{1i} = G_i \gamma, \quad \eta(t_i) = \int_{t_{i-1}}^{t_i} e^{A(t_i - r)} \zeta(dr)$ and $\eta(t_i)$ satisfies $E(\eta(t_i)) = 0_{n\times 1}, E(\eta(t_i)\eta(t_j)') = 0_{n\times n}$ for $i \neq j$ and $E(\eta(t_i)\eta(t_i)') = \Omega_i = \int_0^{\delta_i} e^{Ar} \Sigma e^{A'r} dr, \quad i = 1, \ldots, N.$
The discrete time model with unequally spaced stock data generated by (1) follows a VAR(1)process. In the discrete time model, the coefficients are time-varying and the disturbances are heteroskedastic while the parameters in the continuous time model (equation (1)) are constant and the variance is homoskedastic. These discrepancies are generated by the variations of the sampling intervals.
In the case of unequally spaced flow variables, the observations constitute a sequence of flow vectors of the form

$$x_{t_i} = \int_{t_{i-1}}^{t_i} x(r) dr = \int_0^{\delta_i} x(t_i - r) dr = \int_0^{\delta_i} x(t_{i-1} + r) dr, \quad i = 1, \ldots, T. \tag{4}$$

With equally spaced observations a discrete time representation can be obtained by integrating (3) over the common observation interval. This procedure, however, is inappropriate when the

[2] see details in Appendix

observations are unequally spaced due to the following reason. Integration over $(t_{i-1}, t_i]$ will yield $x_{t_i}$ on the left-hand-side but, on the right-hand-side,

$$\int_{t_{i-1}}^{t_i} x(r-\delta_i)dr = \int_{t_{i-1}-\delta_i}^{t_i-\delta_i} x(s)ds = \int_{t_{i-1}-\delta_i}^{t_{i-1}} x(s)ds \neq x_{t_{i-1}} = \int_{t_{i-2}}^{t_{i-1}} x(s)ds.$$

The problem concerns the lower limit where $t_{i-1} - \delta_i \neq t_{i-2} = t_{i-1} - \delta_{i-1}$. The approach to derive the discrete time representation, which is presented in the previous chapter imposes restrictions on the matrix $A$ to be nonsingular. This rules out applications to systems involving unit roots and cointegration. This section provides the discrete time representation which has the advantage of not requiring any additional conditions beyond Assumption 1. The derivation relies on the following lemma.

**Lemma 1**. $G_i$ is nonsingular for all $i = 1, \ldots, T$.

**Proof**. From the series expansion of $\exp\{As\}$ we find that

$$\begin{aligned} G_i = \int_0^{\delta_i} e^{As} ds &= \int_0^{\delta_i} \sum_{j=0}^{\infty} \frac{A^j s^j}{j!} ds \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \left( \int_0^{\delta_i} s^j ds \right) A^j \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{\delta_i^{j+1}}{j+1} \right) A^j \\ &= \sum_{j=0}^{\infty} c_j A^j \end{aligned}$$

where $c_j = \delta_i^{j+1}/(j+1)!$. It is shown by Abadir and Magnus (2005, p.262) that, if $\Phi(A) = \sum_{j=0}^{\infty} c_j A^j$, then $|\Phi(A)| = \prod_{i=1}^{n} \phi(\lambda_i)$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$ (not necessarily distinct) and $\phi(\lambda) = \sum_{j=0}^{\infty} c_j \lambda^j$. The matrix $G_i$ is clearly of the form $\Phi(A)$ and we shall demonstrate that $|G_i| \neq 0$, using the above result, and, hence, that $G_i$ is nonsingular. Note that, if an eigenvalue of $A$ is zero, then $\phi(0) = c_0 = \delta_i$ whereas, for real or complex $\lambda \neq 0$,

$$\phi(\lambda) = \sum_{j=0}^{\infty} c_j \lambda^j = \sum_{j=0}^{\infty} \frac{\delta_i^{j+1} \lambda^j}{(j+1)!} = \frac{1}{\lambda} \sum_{j=0}^{\infty} \frac{\delta_i^{j+1} \lambda^{j+1}}{(j+1)!} = \frac{1}{\lambda} \int_0^{\delta_i \lambda} e^s ds$$

i.e. $\phi(\lambda) = (e^{\delta_i \lambda} - 1)/\lambda$. Let there be $n_1$ zero eigenvalues and $n_2$ non-zero eigenvalues, where $n_1 + n_2 = n$, ordered so that $\lambda_j = 0$ $(j = 1, \ldots, n_1)$ and $\lambda_j \neq 0$ $(j = n_1 + 1, \ldots, n)$. Then

$$|G_i| = \prod_{j=1}^{n_1} \phi(0) \prod_{j=n_1+1}^{n} \phi(\lambda_j) = \delta_i^{n_1} \prod_{j=n_1+1}^{n} \frac{(e^{\delta_i \lambda_j} - 1)}{\lambda_j}$$

because $\phi(0) = \delta_i$. This expression can be zero only if $\delta_i = 0$ or if $e^{\delta_i \lambda_j} - 1 = 0$. The first possibility is ruled out because $\delta_i > 0$ and the second because $\delta_i \lambda_j \neq 0$ owing to $\delta_i > 0$ and $\lambda_j \neq 0$ for $j = n_1 + 1, \ldots, n$. Hence $|G_i| \neq 0$ and $G_i$ is nonsingular as claimed. End of proof.

The invertibility of $G_i$ is used in the derivation of the exact discrete time model; Lemma 1 shows that no further conditions need to be imposed on the matrix $A$ for this property to hold. The discrete time representation is given by Theorem 1.

**Theorem 1**. Let $x(t)$ be a flow variable generated by (1) which is observed as the sequence in (4). Under Assumption 1, the observations satisfy

$$x_{t_1} = m_{01} + G_1 x(0) + \xi_{t_1},$$

$$x_{t_i} = m_{0i} + m_{1i} t_i + \Phi_i x_{t_{i-1}} + \xi_{t_i}, \quad i = 2, \ldots, T,$$

where $m_{01} = \rho_{01} + \rho_{11}\delta_1$ and, for $i = 2, \ldots, N$, $\Phi_i = G_i F_{i-1} G_{i-1}^{-1}$,

$$m_{0i} = \rho_{0i} + G_i(c_{0,i-1} - c_{1,i-1}\delta_i) - \Phi_i(\rho_{0,i-1} - \rho_{1,i-1}\delta_i),$$

$$m_{1i} = \rho_{1i} + G_i c_{1,i-1} - \Phi_i \rho_{1,i-1},$$

$$\rho_{0i} = J_i \mu + (M_i - K_i - J_i \delta_i)\gamma$$

$$\rho_{1i} = J_i \gamma.$$

Furthermore, $\xi_{t_i}$ is a heteroskedastic MA(1) process with autocovariance matrices given by

$$\Omega_{0,i} = E[\xi_{t_i}\xi_{t_i}'] = \begin{cases} \int_0^{\delta_1} G(s)\Sigma G(s)' ds, & i = 1, \\ \int_0^{\delta_i} G(s)\Sigma G(s)' ds + \int_0^{\delta_{i-1}} \Gamma_i(s)\Sigma\Gamma_i(s)' ds, & i = 2, \ldots, T, \end{cases}$$

$$\Omega_{-1,i} = E[\xi_{t_i}\xi_{t_{i-1}}'] = \int_0^{\delta_{i-1}} \Gamma_i(s)\Sigma G(s)' ds, \quad i = 2, \ldots, T,$$

$$\Omega_{1,i} = E[\xi_{t_i}\xi_{t_{i+1}}'] = \int_0^{\delta_i} G(s)\Sigma\Gamma_{i+1}(s)' ds, \quad i = 1, \ldots, T-1,$$

where $\Gamma_i(x) = G_i F(x) - \Phi_i G(x)$.

**Proof**. We first derive the equations for $i = 2, \ldots, N$ and then for $i = 1$. (4) implies that

$$x(t_{i-1} + s) = c_s + e^{As} x(t_{i-1}) + \int_{t_{i-1}}^{t_{i-1}+s} e^{A(t_{i-1}+s-r)}\zeta(dr), 0 < s < \delta_i \tag{5}$$

where

$$c_s = \int_{t_{i-1}}^{t_{i-1}+s} e^{A(t_{i-1}+s-r)} (\mu + \gamma r)\, dr.$$

Evaluating this deterministic integral enables us to show that

$$c_s = G(s)\mu - H(s)\gamma + G(s)\gamma(t_{i-1} + s).$$

Hence integrating (5) over $s \in (0, \delta_i]$ results in

$$\begin{aligned}\int_0^{\delta_i} x(t_{i-1}+s)ds &= \int_0^{\delta_i} G(s)ds\mu - \int_0^{\delta_i} H(s)ds\gamma + \int_0^{\delta_i} G(s)(t_{i-1}+s)ds\gamma \\ &+ \left(\int_0^{\delta_i} e^{As}ds\right) x(t_{i-1}) + \int_0^{\delta_i} \int_{t_{i-1}}^{t_{i-1}+s} e^{A(t_{i-1}+s-r)}\zeta(dr)ds.\end{aligned}$$

Given that $t_{i-1} = t_i - \delta_i$, the above equation can be written as

$$x_{t_i} = \rho_{0i} + \rho_{1i}t_i + G_i x(t_{i-1}) + e_{t_i}, \quad i = 1, \ldots, T, \tag{6}$$

where $\rho_{0i} = J_i\mu + (M_i - K_i - J_i\delta_i)\gamma, \rho_{1i} = J_i\gamma$, and $e_{t_i} = \int_0^{\delta_i} \int_{t_{i-1}}^{t_{i-1}+s} e^{A(t_{i-1}+s-r)}\zeta(dr)ds$. Using Lemma 1 we can solve (6) for $x(t_{i-1})$:

$$x(t_{i-1}) = G_i^{-1}\left(x_{t_i} - \rho_{0i} - \rho_{1i}t_i - e_{t_i}\right). \tag{7}$$

But, from (3), we know that

$$x(t_{i-1}) = c_{0,i-1} + c_{1,i-1}t_{i-1} + F_{i-1}x(t_{i-2}) + \eta_{t_{i-1}}. \tag{8}$$

Using (7) and its lag to substitute for $x(t_{i-1})$ and $x(t_{i-2})$ in (8) results in

$$\begin{aligned}G_i^{-1}\left(x_{t_i} - \rho_{0i} - \rho_{1i}t_i - e_{t_i}\right) &= c_{0,i-1} + c_{1,i-1}t_{i-1} \\ &+ F_{i-1}G_{i-1}^{-1}\left(x_{t_{i-1}} - \rho_{0,i-1} - \rho_{1,i-1}t_{i-1} - e_{t_{i-1}}\right) + \eta_{t_{i-1}}.\end{aligned} \tag{9}$$

Multiplying (9) by $G_i$, using $t_{i-1} = t_i - \delta_i$, we obtain

$$\begin{aligned}x_{t_i} &= m_{0i} + m_{1i}t_i + \Phi_i x_{t_{i-1}} + \xi_{t_i}, \quad i = 1, \ldots, T, \\ where \Phi_i &= G_iF_{i-1}G_{i-1}^{-1}, m_{0i} = \rho_{0i} + G_i(c_{0,i-1} - c_{1,i-1}\delta_i) - \Phi_i(\rho_{0,i-1} - \rho_{1,i-1}\delta_i), \\ m_{1i} &= \rho_{1i} + G_ic_{1,i-1} - \Phi_i\rho_{1,i-1} \; and \; \xi_{t_i} = e_{t_i} - \Phi_i e_{t_{i-1}} + G_i\eta_{t_{i-1}}.\end{aligned} \tag{10}$$

The equation for $i = 1$ is obtained in a similar manner; setting $t_{i-1} = 0$ in (6), and noting that $t_1 = \delta_1$, we obtain

$$x_{t_1} = m_{01} + G_1 x(0) + \xi_{t_1},$$

where $m_{01} = \rho_{01} + \rho_{11}\delta_1$ and $\xi_{t_1} = e_{t_1}$.

To derive properties of the disturbances, it is necessary to reduce the double integral defining $e_{t_i}$ to a more convenient form:

$$\begin{aligned} e_{t_i} &= \int_{t_{i-1}}^{t_i} \left( \int_{r-t_{i-1}}^{\delta_i} e^{A(t_{i-1}+s-r)} ds \right) \zeta(dr) \\ &= \int_{t_{i-1}}^{t_i} \left( \int_0^{t_i - r} e^{Aw} dw \right) \zeta(dr) \\ &= \int_{t_{i-1}}^{t_i} G(t_i - r)\zeta(dr), \quad i = 1, \ldots, T. \end{aligned}$$

Hence, using (10), for $i = 2, \ldots, T$, $\xi_{t_i}$ can be written as

$$\begin{aligned} \xi_{t_i} &= \int_{t_{i-1}}^{t_i} G(t_i - r)\zeta(dr) - \Phi_i \int_{t_{i-2}}^{t_{i-1}} G(t_{i-1} - r)\zeta(dr) + G_i \int_{t_{i-2}}^{t_{i-1}} F(t_{i-1} - r)\zeta(dr) \\ &= \int_{t_{i-1}}^{t_i} G(t_i - r)\zeta(dr) + \int_{t_{i-2}}^{t_{i-1}} \Gamma_i(t_{i-1} - r)\zeta(dr), \end{aligned}$$

where $\Gamma_i(x) = G_i F(x) - \Phi_i G(x)$, while for $i = 1$ we have

$$\xi_{t_1} = \int_0^{t_1} G(t_1 - r)\zeta(dr).$$

The autocovariances follow from these expressions. Properties of $\xi$ are obtained as

$$E[\xi_{t_i}] = 0, i = 1, \ldots, T,$$

$$\begin{aligned} E[\xi_{t_1}\xi_{t_1}{}'] &= E\left[\int_0^{t_1} G(t_1 - r)\zeta(dr)\right]\left[\int_0^{t_1} G(t_1 - r)\zeta(dr)\right]' \\ &= \int_0^{t_1} G(t_1 - r)\Sigma G(t_1 - r)' dr \\ &= \int_0^{\delta_1} G(s)\Sigma G(s)' ds, i = 1, \end{aligned}$$

$$\begin{aligned} E[\xi_{t_i}\xi_{t_i}{}'] &= E\left[\int_{t_{i-1}}^{t_i} G(t_i - r)\zeta(dr)\right]\left[\int_{t_{i-1}}^{t_i} G(t_i - r)\zeta(dr)\right]' \\ &+ E\left[\int_{t_{i-2}}^{t_{i-1}} \Gamma_i(t_{i-1} - r)\zeta(dr)\right]\left[\int_{t_{i-2}}^{t_{i-1}} \Gamma_i(t_{i-1} - r)\zeta(dr)\right]' \\ &= \int_{t_{i-1}}^{t_i} G(t_i - r)G(t_i - r)' dr + \int_{t_{i-2}}^{t_{i-1}} \Gamma_i(t_{i-1} - r)\Sigma\Gamma_i(t_{i-1} - r)' dr \\ &= \int_0^{\delta_i} G(s)\Sigma G(s)' ds + \int_0^{\delta_{i-1}} \Gamma_i(s)\Sigma\Gamma_i(s)' ds, i = 2, \ldots, T, \end{aligned}$$

$$
\begin{aligned}
E[\xi_{t_i}\xi_{t_{i-1}}'] &= E\left[\int_{t_{i-2}}^{t_{i-1}} \Gamma_i(t_{i-1}-r)\zeta(dr)\right]\left[\int_{t_{i-2}}^{t_{i-1}} G(t_{i-1}-r)\zeta(dr)\right]' \\
&= \int_{t_{i-2}}^{t_{i-1}} \Gamma_i(t_{i-1}-r)\Sigma G(t_{i-1}-r)'dr \\
&= \int_0^{\delta_{i-1}} \Gamma_i(s)\Sigma G(s)'ds, i = 2, \ldots, T,
\end{aligned}
$$

$$
\begin{aligned}
E[\xi_{t_i}\xi_{i+1}{}'] &= E\left[\int_{t_{i-1}}^{t_i} G(t_i-r)\zeta(dr)\right]\left[\int_{t_{i-1}}^{t_i} \Gamma_i(t_i-r)\zeta(dr)\right] \\
&= \int_{t_{i-1}}^{t_i} G(t_i-r)\Sigma\Gamma_i(t_i-r)'dr \\
&= \int_0^{\delta_i} G(s)\Sigma\Gamma_i(s)'ds, i = 1, \ldots, T-1.
\end{aligned}
$$

End of Proof.

Theorem 1 shows that the discrete time model with flow variables follows a VARMA(1, 1) process with time-varying coefficients and heteroskedasticity. The heteroskedastic variances arises due to the variations in the sampling intervals while the heteroskedastic MA(1) disturbances arise due to the flow nature of the observations. Furthermore, Theorem 1 does not require restrictions on the matrix A (i.e. requiring A to be nonsingular), which indicates that the results of the theorem are applicable in nonstationary and cointegrated models as well as stationary systems. In addition, Theorem 1 can be used when data are equally spaced, namely, when $\delta_i = 1$ for all $i$. The advantage of this model is that A is not required to be nonsingular, which hence does not rule out applications to nonstationary systems.

# 3 AN EXACT DISCRETE TIME MODEL WITH MIXED SAMPLES

In this section, a system that includes both stock and flow variables is considered. The derivation of the exact discrete time representation follows Agbeyegbe's (1987) procedure. In the case of mixed samples, both the stocks and flows are assumed to be observed at the same frequency (same unequally-spaced points in time). The observations are of the form

$$
x(t_i) = \begin{bmatrix} x^s(t_i) \\ x^f(t_i) \end{bmatrix} = \begin{bmatrix} x^s(t_i) \\ \int_{t_{i-1}}^{t_i} x^f(r)dr \end{bmatrix}, i = 1, 2, \cdots, T. \tag{11}
$$

$x^s(t_i)$ is a vector of $(n^s \times 1)$ stock variables and $x^f(t_i)$ is a vector of $(n^f \times 1)$ flow variables, with $n^s + n^f = n$. The system of stock and flow variables, generated by (1), is partitioned as

$$d(x^s(t)) = [A^{ss}x^s(t) + A^{sf}x^f(t) + \mu^s + \gamma^s t]dt + \zeta^s(dt), \tag{12}$$

$$d(x^f(t)) = [A^{fs}x^s(t) + A^{ff}x^f(t) + \mu^f + \gamma^f t]dt + \zeta^f(dt), \tag{13}$$

where $A = \begin{bmatrix} A^{ss} & A^{sf} \\ A^{fs} & A^{ff} \end{bmatrix}$, $\mu = \begin{bmatrix} \mu^s \\ \mu^f \end{bmatrix}$, $\gamma = \begin{bmatrix} \gamma^s \\ \gamma^f \end{bmatrix}$,and $\zeta(dt) = \begin{bmatrix} \zeta^s(dt) \\ \zeta^f(dt) \end{bmatrix}$.

In order for Theorem 2 to be valid, we shall need the following assumption on the sub-matrix of $A$

**Assumption 2**. The sub-matrix $A^{ss}$ is non-singular.

The main challenge with mixed data is eliminating unobservable terms from the system: integrals of stock variables, $\int_{t_{i-1}}^{t_i} x^s(r)dr$, and the levels of flow variables, $x^f(t_i)$. To derive the exact discrete time model, it is necessary to define an $(n \times 1)$ random vector $z_{t_1}, z_{t_2}, \cdots, z_{t_n}$ in the form

$$z_{t_i} = \begin{bmatrix} x^s(t_i) - x^s(t_{i-1}) \\ \int_{t_{i-1}}^{t_i} x^f(r)\, dr \end{bmatrix}, i = 1, 2, \cdots, T. \tag{14}$$

The vector $z_{t_i}$ defined above represents a mixture of stock variables and flow variables. The exact discrete time model for mixed data is given by Theorem 2.

**Theorem 2**. Let $x(t)$ be generated by (1) which is observed as the mixed-sample sequence in (11). Under Assumption 1 and 2, the random vectors $z_{t_1}, z_{t_2}, \cdots, z_{t_n}$ defined by (14) satisfy the system

$$z_{t_i} = \Pi_i z_{t_{i-1}} + g_i + \epsilon_{t_i}, \tag{15}$$

$$E[\epsilon_{t_i}] = 0,$$

$$\begin{aligned} V_i &= E[\epsilon_{t_i}\epsilon_{t_i}'] \\ &= \begin{cases} \int_0^{\delta_1} \Psi(s)\Sigma\Psi(s)'\, ds & i = 1, \\ \int_0^{\delta_i} \Psi(s)\Sigma\Psi(s)'\, ds + \int_0^{\delta_{i-1}} S(s)\Sigma S(s)'\, ds, & i = 2, \cdots, T, \end{cases} \end{aligned}$$

$$\begin{aligned} &W_{-1,i} = \mathrm{E}[\epsilon_{t_i}\epsilon_{t_1}'] = \int_0^{\delta_{i-1}} S(s)\Sigma\Psi(s)'\, ds \qquad i = 2, \cdots, T, \\ &W_i = \int_0^{\delta_i} \Psi(s)\Sigma S(s)'\, ds \qquad i = 1, \cdots, T-1, \end{aligned}$$

where

$$\Pi_i = \begin{bmatrix} \Pi_i^{ss} & \Pi_i^{sf} \\ \Pi_i^{fs} & \Pi_i^{ff} \end{bmatrix},$$

$$g_i = \begin{bmatrix} g_i^s \\ g_i^f \end{bmatrix},$$

$$\begin{aligned} \epsilon_{t_i} &= \int_{t_{i-1}}^{t_i} \Psi(t_i - r)\zeta(dr) + \int_{t_{i-2}}^{t_{i-1}} S(t_{i-1} - r)\zeta(dr) \\ &= \begin{bmatrix} \epsilon_{t_i}^s \\ \epsilon_{t_i}^f \end{bmatrix}, \end{aligned}$$

$$\Pi_i^{ss} = [A^{ss}\Phi_i^{ss} + A^{sf}\Phi_i^{fs}][A^{ss}]^{-1},$$

$$\Pi_i^{sf} = [A^{ss}\Phi_i^{sf} + A^{sf}\Phi_i^{ff}] - \Pi_i^{11}A^{sf},$$

$$\Pi_i^{fs} = \Phi_i^{fs}[A^{ss}]^{-1},$$

$$\Pi_i^{ff} = \Phi_i^{ff} - \Pi_i^{21}A^{sf},$$

$$g_i^s = A^{ss}m_{0i}^s + A^{sf}m_{0i}^f + (A^{ss}m_{1i}^s + A^{sf}m_{1i}^f)t_i + \int_{t_{i-1}}^{t_i} [\mu^s + \gamma^s r]\,dr - \Pi_i^{ss}\int_{t_{i-2}}^{t_{i-1}} [\mu^s + \gamma^s r]\,dr,$$

$$g_i^f = m_{0i}^f + m_{1i}^f t_i - \Pi_i^{fs}\int_{t_{i-2}}^{t_{i-1}} [\mu^s + \gamma^s r]\,dr,$$

$$\epsilon_{t_i}^s = \int_{t_{i-1}}^{t_i} \zeta^s(dr) + A^{ss}\xi_{t_i}^s + A^{sf}\xi_{t_i}^f - \Pi_i^{ss}\int_{t_{i-2}}^{t_{i-1}} \zeta^s(dr),$$

$$\epsilon_{t_i}^f = \xi_{t_i}^f - \Pi_i^{fs}\int_{t_{i-2}}^{t_{i-1}} \zeta^s(dr),$$

$$\Psi(t_i - r) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A^{ss} & A^{sf} \\ 0 & I \end{bmatrix} \begin{bmatrix} [G(t_i - r)]^{ss} & [G(t_i - r)]^{sf} \\ [G(t_i - r)]^{fs} & [G(t_i - r)]^{ff} \end{bmatrix},$$

$$S(t_{i-1} - r) = \begin{bmatrix} A^{ss} & A^{sf} \\ 0 & I \end{bmatrix} \begin{bmatrix} [\Gamma_i(t_{i-1} - r)]^{ss} & [\Gamma_i(t_{i-1} - r)]^{sf} \\ [\Gamma_i(t_{i-1} - r)]^{fs} & [\Gamma_i(t_{i-1} - r)]^{ff} \end{bmatrix} - \begin{bmatrix} \Pi_i^{ss} & 0 \\ \Pi_i^{fs} & 0 \end{bmatrix},$$

$$G(t_i - r) = \begin{bmatrix} [G(t_i - r)]^{ss} & [G(t_i - r)]^{sf} \\ [G(t_i - r)]^{fs} & [G(t_i - r)]^{ff} \end{bmatrix},$$

$$\Gamma_i(t_{i-1}-r) \quad = \quad \begin{bmatrix} [\Gamma_i(t_{i-1}-r)]^{ss} & [\Gamma_i(t_{i-1}-r)]^{sf} \\ [\Gamma_i(t_{i-1}-r)]^{fs} & [\Gamma_i(t_{i-1}-r)]^{ff} \end{bmatrix}.$$

**Proof**. Integrating (1) over the interval $[t_{i-1}, t_i]$ obtains

$$x(t_i) - x(t_{i-1}) = A\int_{t_{i-1}}^{t_i} x(r)\,dr + \int_{t_{i-1}}^{t_i} [\mu + \gamma r]\,dr + \int_{t_{i-1}}^{t_i} \zeta(dr), \tag{16}$$

while the first row of equation (16) is

$$x^s(t_i) - x^s(t_{i-1}) = A^{ss}\int_{t_{i-1}}^{t_i} x^s(r)\,dr + A^{sf}\int_{t_{i-1}}^{t_i} x^f(r)\,dr + \int_{t_{i-1}}^{t_i} [\mu^s + \gamma^s r]\,dr + \int_{t_{i-1}}^{t_i} \zeta^s(dr). \tag{17}$$

Partitioning (10) as

$$\int_{t_{i-1}}^{t_i} x^s(r)\,dr = {\Phi_i}^{ss}\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr + {\Phi_i}^{sf}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + {m_{0i}}^s + {m_{1i}}^s t_i + {\xi_{ti}}^s, \tag{18}$$

$$\int_{t_{i-1}}^{t_i} x^f(r)\,dr = {\Phi_i}^{fs}\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr + {\Phi_i}^{ff}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + {m_{0i}}^f + {m_{1i}}^f t_i + {\xi_{ti}}^f, \tag{19}$$

where

$$\Phi_i \quad = \quad \begin{bmatrix} {\Phi_i}^{ss} & {\Phi_i}^{sf} \\ {\Phi_i}^{fs} & {\Phi_i}^{ff} \end{bmatrix},$$

$$m_{0i} \quad = \quad \begin{bmatrix} {m_{0i}}^s \\ {m_{0i}}^f \end{bmatrix},$$

$$m_{1i} \quad = \quad \begin{bmatrix} {m_{1i}}^s \\ {m_{1i}}^f \end{bmatrix},$$

and

$$\xi_{ti} \quad = \quad \begin{bmatrix} {\xi_{ti}}^s \\ {\xi_{ti}}^f \end{bmatrix}.$$

Substituting out $\int_{t_{i-1}}^{t_i} x^s(r)\,dr$ and $\int_{t_{i-1}}^{t_i} x^f(r)\,dr$ in (17) by (18) and (19), respectively

$$\begin{aligned} x^s(t_i) - x^s(t_{i-1}) &= [A^{ss}\Phi_i{}^{ss} + A^{sf}\Phi_i{}^{fs}] \int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr + [A^{ss}\Phi_i{}^{sf} + A^{sf}\Phi_i{}^{ff}] \int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr \\ &+ A^{ss}m_{0i}{}^s + A^{sf}m_{0i}{}^f + [A^{ss}m_{1i}{}^s + A^{sf}m_{1i}{}^f]t_i + \int_{t_{i-1}}^{t_i} [\mu^s + \gamma^s r]\,dr \\ &+ A^{ss}\xi_{ti}{}^s + A^{sf}\xi_{ti}{}^f + \int_{t_{i-1}}^{t_i} \zeta^s(dr). \end{aligned} \tag{20}$$

From (17) we obtain

$$\begin{aligned} \int_{t_{i-1}}^{t_i} x^s(r)\,dr &= [A^{ss}]^{-1}[x^s(t_i) - x^s(t_{i-1})] - [A^{ss}]^{-1}A^{sf} \int_{t_{i-1}}^{t_i} x^f(r)\,dr \\ &- [A^{ss}]^{-1} \int_{t_{i-1}}^{t_i} [\mu^s + \gamma^s r]\,dr - [A^{ss}]^{-1} \int_{t_{i-1}}^{t_i} \zeta^s(dr). \end{aligned} \tag{21}$$

Lagging (21) for one period

$$\begin{aligned} \int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr &= [A^{ss}]^{-1}[x^s(t_{i-1}) - x^s(t_{i-2})] - [A^{ss}]^{-1}A^{sf} \int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr \\ &- [A^{ss}]^{-1} \int_{t_{i-2}}^{t_{i-1}} [\mu^s + \gamma^s r]\,dr - [A^{ss}]^{-1} \int_{t_{i-2}}^{t_{i-1}} \zeta^s(dr). \end{aligned} \tag{22}$$

The object now is to eliminate the unobservale term, $\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr$ in (20) and (19). Substituting out $\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr$ in (20) using (22)

$$\begin{aligned} x^s(t_i) - x^s(t_{i-1}) &= [A^{ss}\Phi_i{}^{ss} + A^{sf}\Phi_i{}^{fs}][A^{ss}]^{-1} \left\{ [x^s(t_{i-1}) - x^s(t_{i-2})] - A^{sf} \int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr \right\} \\ &+ [A^{ss}\Phi_i{}^{sf} + A^{sf}\Phi_i{}^{ff}] \int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + A^{ss}m_{0i}{}^s + A^{sf}m_{0i}{}^f \\ &+ [A^{ss}m_{1i}{}^s + A^{sf}m_{1i}{}^f]t_i + \int_{t_{i-1}}^{t_i} [\mu^s + \gamma^s r]\,dr \\ &- [A^{ss}\Phi_i{}^{ss} + A^{sf}\Phi_i{}^{fs}][A^{ss}]^{-1} \int_{t_{i-2}}^{t_{i-1}} [\mu^s + \gamma^s r]\,dr \\ &+ A^{ss}\xi_{ti}{}^s + A^{sf}\xi_{ti}{}^f + \int_{t_{i-1}}^{t_i} \zeta^s(dr) \\ &- [A^{ss}\Phi_i{}^{ss} + A^{sf}\Phi_i{}^{fs}][A^{ss}]^{-1} \int_{t_{i-2}}^{t_{i-1}} \zeta^s(dr) \\ &= \Pi_i{}^{ss}[x^s(t_{i-1}) - x^s(t_{i-2})] + \Pi_i{}^{sf} \int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + g_i{}^s + \epsilon_{ti}{}^s. \end{aligned} \tag{23}$$

Substituting out $\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr$ in (19) using (22)

$$\begin{aligned}
\int_{t_{i-1}}^{t_i} x^f(r)\,dr &= {\Phi_i}^{fs}[A^{ss}]^{-1}\left\{[x^s(t_{i-1})-x^s(t_{i-2})]-A^{sf}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr\right\} \\
&+ {\Phi_i}^{ff}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + {m_{0i}}^f + {m_{1i}}^f t_i \\
&- {\Phi_i}^{fs}[A^{ss}]^{-1}\int_{t_{i-2}}^{t_{i-1}}[\mu^s+\gamma^s r]\,dr + {\xi_{ti}}^f - {\Phi_i}^{fs}[A^{ss}]^{-1}\int_{t_{i-2}}^{t_{i-1}}\zeta^s(dr) \\
&= {\Pi_i}^{fs}[x^s(t_{i-1})-x^s(t_{i-2})] + {\Pi_i}^{ff}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + {g_i}^f + {\epsilon_{ti}}^f. \qquad (24)
\end{aligned}$$

Combining (23) and (24) we obtain (15)

$$\begin{bmatrix} x^s(t_i)-x^s(t_{i-1}) \\ \int_{t_{i-1}}^{t_i} x^f(r)\,dr \end{bmatrix} = \begin{bmatrix} \Pi_i^{ss} & \Pi_i^{sf} \\ \Pi_i^{fs} & \Pi_i^{ff} \end{bmatrix}\begin{bmatrix} x^s(t_{i-1})-x^s(t_{i-2}) \\ \int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr \end{bmatrix} + \begin{bmatrix} g_i^s \\ g_i^f \end{bmatrix} + \begin{bmatrix} \epsilon_{t_i}^s \\ \epsilon_{t_i}^f \end{bmatrix}.$$

Properties of vector $\epsilon_{t_i}$ depend on properties of the continuous time disturbance vector $\zeta(dt)$. Mean of $\epsilon_{t_i}$:

$$E[\epsilon_{t_i}] = 0, i = 1,\ldots,T.$$

Variance of $\epsilon_{t_i}$:

$$\begin{aligned}
E[\epsilon_{t_1}\epsilon_{t_1}'] &= E\left[\int_0^{t_1}\Psi(t_1-r)\zeta(dr)\right]\left[\int_0^{t_1}\Psi(t_1-r)\zeta(dr)\right]' \\
&= \int_0^{t_1}\Psi(t_1-r)\Sigma\Psi(t_1-r)'dr \\
&= \int_0^{\delta_1}\Psi(s)\Sigma\Psi(s)'ds, i=1,
\end{aligned}$$

$$\begin{aligned}
E[\epsilon_{t_i}\epsilon_{t_i}'] &= E\left[\int_{t_{i-1}}^{t_i}\Psi(t_i-r)\zeta(dr)\right]\left[\int_{t_{i-1}}^{t_i}\Psi(t_i-r)\zeta(dr)\right]' \\
&+ E\left[\int_{t_{i-2}}^{t_{i-1}}S(t_{i-1}-r)\zeta(dr)\right]\left[\int_{t_{i-2}}^{t_{i-1}}S(t_{i-1}-r)\zeta(dr)\right]' \\
&= \int_{t_{i-1}}^{t_i}\Psi(t_i-r)\Sigma\Psi(t_i-r)'\,dr + \int_{t_{i-2}}^{t_{i-1}}S(t_{i-1}-r)\Sigma S'(t_{i-1}-r)\,dr \\
&= \int_0^{\delta_i}\Psi(s)\Sigma\Psi(s)'\,ds + \int_0^{\delta_{i-1}}S(s)\Sigma S(s)'\,ds, i=2,\ldots,T.
\end{aligned}$$

Autocorariance of $\epsilon_{t_i}$:

$$\begin{aligned} E[\epsilon_{t_i}\epsilon'_{t_{i-1}}] &= E\left[\int_{t_{i-2}}^{t_{i-1}} S(t_{i-1}-r)\zeta(dr)\right]\left[\int_{t_{i-2}}^{t_{i-1}} \Psi(t_{i-1}-r)\zeta(dr)\right]' \\ &= \int_{t_{i-2}}^{t_{i-1}} S(t_{i-1}-r)\Sigma\Psi(t_{i-1}-r)'dr \\ &= \int_0^{\delta_{i-1}} S(s)\Sigma\Psi(s)'\,ds, i=2,\ldots,T, \end{aligned}$$

$$\begin{aligned} E[\epsilon_{t_i}\epsilon'_{t_{i+1}}] &= E\left[\int_{t_{i-1}}^{t_i} \Psi(t_i-r)\zeta(dr)\right]\left[\int_{t_{i-1}}^{t_i} S(t_i-r)\zeta(dr)\right]' \\ &= \int_{t_{i-1}}^{t_i} \Psi(t_i-r)\Sigma S(t_i-r)'dr \\ &= \int_0^{\delta_i} \Psi(s)\Sigma S(s)'\,ds, i=1,\ldots,T-1. \end{aligned}$$

End of proof.

Theorem 2 shows that, with mixed data, the exact discrete time model follows a VARMA(1,1) process with time-varying coefficient and disturbance vector $\epsilon_{ti}$ is a heteroskedastic MA(1). The underlying continuous time model, instead, has constant parameters and homoskedastic variance. Moreover, the autocovariances are asymmetric due to the (possibly asymmetric) variations in the length of sampling intervals.

# 4 SIMULATION EVIDENCE

A Monte Carlo simulation is conducted to examine the performance of estimation of continuous time models where unequal sampling intervals are correctly measured. This study considered a cointegrated system of flow variables whose sampling intervals coincide with the variation of calendar months. The lengths of monthly sampling intervals vary from 28 days to 31 days, which are normalised by dividing each interval by 30. Note that we ignore leap years, assuming each February has 28 days for reducing computation cost. The resulting sampling intervals are $\delta_{\min} = 0.9\dot{3}$, 1.00 and $\delta_{\max} = 1.0\dot{3}$. The model of interest is

$$dx(t) = Ax(t)dt + \zeta(dt), \quad t > 0,$$

where $A = \begin{bmatrix} \alpha_1 & -\alpha_1\beta \\ \alpha_2 & -\alpha_2\beta \end{bmatrix}$ is an $n \times n$ coefficient matrix with $\beta = 1$ and $\alpha_1 - \alpha_2\beta < 0$ and $\zeta(dt)$ satisfy Assumption 1.

The observations are made at points $t_i$ with $t_i = t_{i-1} - \delta_i, i = 2, \cdots, T$ and $\delta_i$ denotes sample intervals. $T$ is the sample size and, specifically, we assume $t_0 = 0$ and $x(0) = 0$ as the Boundary Condition. In this case we have 2 variables hence $n = 2$.

To explore the impact of values of the parameters and sample size on the estimation results, we compare the simulation results with $\alpha_1$ and $\alpha_2$ are -1.25 and 0.75 respectively to the results with $\alpha_1$ and $\alpha_2$ are -0.95 and -0.05 respectively; while the sample size change from 120 (10 year span) to 240 (20 year span). $\beta = 1$ and $\sigma^2 = 0.25$ in all cases.

Using results from Theorem 2.1, the discrete time model is obtained as

$$x_{t_1} = G_1 x(0) + \xi_{t_1},$$

$$x_{t_i} = \Phi_i x_{t_{i-1}} + \xi_{t_i}, \quad i = 2, \ldots, T,$$

where properties of $\xi_{t_i}$ satisfy Theorem 1. The parameters to be estimated are $\theta = [\alpha_1, \alpha_2, \beta, \sigma^2]'$. Estimates of $\theta$ are obtained when the Gaussian log-likelihood function is maximised.

$$L(\theta) = -\frac{T}{2} ln 2\pi - \frac{1}{2} ln|\Omega| - \frac{1}{2}\xi'\Omega^{-1}\xi,$$

where $\xi = [\xi_1{}', \cdots, \xi_T{}']'$ is an $nT \times 1$ vector of disturbances and the $nT \times nT$ covariance matrix of $\xi$ is

$$\begin{aligned} \Omega &= E\left[\xi\xi'\right] \\ &= \begin{bmatrix} \Omega_{0,1} & \Omega_{1,1} & 0 & 0 & \cdots & \cdots & 0 \\ \Omega_{-1,2} & \Omega_{0,2} & \Omega_{1,2} & 0 & \cdots & \cdots & 0 \\ 0 & \Omega_{-1,3} & \Omega_{0,3} & \Omega_{1,3} & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & \Omega_{-1,T-1} & \Omega_{0,T-1} & \Omega_{1,T-1} \\ 0 & 0 & \cdots & \cdots & 0 & \Omega_{-1,T} & \Omega_{0,T} \end{bmatrix}. \end{aligned}$$

Note that estimating the parameters, $\theta$, by maximising the above log likelihood function may not be convenient since inverting the matrix $\Omega$ is (computationally) costly. An alternative method is to find the Choleskey factorization of $\Omega$, then follow a recursive procedure that avoids directly inverting $\Omega$ (see Bergstrom, 1985, 1990).

Let $M$ be the real $nT \times nT$ lower triangular matrix with positive elements along the diagonal such that $MM' = \Omega$. Thus $|\Omega| = |MM'| = |M||M| = |M|^2$ and $\Omega^{-1} = (M')^{-1}(M)^{-1}$. The sub-matrices of $M$, $M_{11}, \cdots, M_{t,t-1}, M_{tt} (t = 2, \cdots, T)$ can be computed as

$$M_{11}M'_{11} = \Omega_{00},$$

$$M_{i,i-1} = \Omega'_{1,i-1}(M'_{i-1,i-1})^{-1},$$

$$M_{i,i}M'_{i,i} = \Omega_{0,i} - M_{i,i-1}M'_{i,i-1}, i = 2, \cdots, T.$$

Then define a normalised $nT \times 1$ vector $\epsilon$, satisfying $E[\epsilon] = 0$ and $E[\epsilon\epsilon'] = I$, such that $M\epsilon = \xi$. Hence we have $\xi'\Omega^{-1}\xi = \xi'(M')^{-1}(M)^{-1}\xi = \epsilon'\epsilon$. Then log-likelihood function can thus be evaluated as

$$L = \sum_{i=1}^{nT} ({\epsilon_i}^2 + 2ln(m_{ii})),$$

where $m_{ii}$ is the i-th diagonal element of $M$.

Then $\xi$ can be computed recursively as

$$\xi_1 = M_{11}\epsilon_1,$$

$$\xi_i = M_{i,i-1}\epsilon_{i-1} + M_{i,i}\epsilon_i, i = 2, \cdots, T.$$

The Gaussian estimates of $\theta$ are obtained when L is minimised. See appendix for derivation details. With the simulated unequally-spaced data, we re-estimated the parameters using the model, which sampling intervals are treated as equal ("equally-spaced" model) and are normalised as unity. Namely, $\delta = t - (t - 1) = 1$ for all observations. The estimation procedure is very similar to the model of interest ("unequally-spaced" model). See appendix for derivations. We then compared the estimations results from using the two models, expecting the estimates of "unequally-spaced" model to have smaller estimation bias.

The results from $10,000$ replications in each case are presented in Table 1. The table contains the simulation bias (calculated as estimated value minus fixed value) and standard error for each estimator (in the parenthesis under). The estimates of $\alpha_1, \alpha_2, \beta$ and $\sigma^2$ are denoted by $\hat{\alpha_1}, \hat{\alpha_2}, \hat{\beta}$ and $\hat{\sigma^2}$, respectively. "Model I" indicates the "unequally-spaced" model while "Model II" indicates the "equally-spaced" model. The estimation bias for Model I estimates are smaller, in absolute terms, than for Model II estimates, except for estimates of $\sigma^2$. The bias in estimates of $\sigma^2$ is smaller for Model II, though the standard errors of these estimates are slightly larger for Model II. The standard errors are smaller for Model I than for Model II in all cases. Moreover, estimation bias, for both models, get smaller with the increase in sample size. Interestingly, the bias of estimates of $\beta$ is of different signs in the 2 different parameter configurations. Overall, the results are broadly favouring Model I, which accounts for the unequal sampling intervals, suggesting that there are improvements in the estimate properties when the sampling intervals are correctly measured.

Table 1: Monte Carlo Simulation Results

| | Parameter | $\alpha_1$ | $\alpha_2$ | $\beta$ | $\sigma^2$ |
|---|---|---|---|---|---|
| | Fixed Value | $-1.25$ | 0.75 | 1 | 0.25 |
| $T = 120$ | Estimate | $\hat{\alpha_1}$ | $\hat{\alpha_2}$ | $\hat{\beta}$ | $\hat{\sigma^2}$ |
| | Bias (Model I) | $-0.022$ | 0.035 | 0.033 | $-0.127$ |
| | | (0.0665) | (0.0762) | (0.0749) | (0.0006) |
| | Bias (Model II) | 0.040 | 0.046 | 0.040 | $-0.122$ |
| | | (0.0716) | (0.0828) | (0.0862) | (0.0007) |
| $T = 240$ | Bias (Model I) | $-0.014$ | 0.018 | 0.012 | $-0.125$ |
| | | (0.0291) | (0.0348) | (0.0276) | (0.0003) |
| | Bias (Model II) | $-0.031$ | 0.030 | 0.016 | $-0.120$ |
| | | (0.0316) | (0.0393) | (0.0339) | (0.0003) |
| | Parameter | $\alpha_1$ | $\alpha_2$ | $\beta$ | $\sigma^2$ |
| | Fixed Value | $-0.95$ | $-0.05$ | 1 | 0.25 |
| $T = 120$ | Estimate | $\hat{\alpha_1}$ | $\hat{\alpha_2}$ | $\hat{\beta}$ | $\hat{\sigma^2}$ |
| | Bias (Model I) | $-0.042$ | 0.043 | $-0.016$ | $-0.122$ |
| | | (0.026) | (0.0181) | (0.0283) | (0.0007) |
| | Bias (Model II) | $-0.061$ | 0.052 | $-0.016$ | $-0.115$ |
| | | (0.0277) | (0.0202) | (0.0332) | (0.0009) |
| $T = 240$ | Bias (Model I) | $-0.019$ | 0.021 | $-0.011$ | $-0.123$ |
| | | (0.0104) | (0.007) | (0.0117) | (0.0003) |
| | Bias (Model II) | $-0.040$ | 0.027 | $-0.013$ | $-0.114$ |
| | | (0.0113) | (0.0087) | (0.0156) | (0.0005) |

# 5 CONCLUSION

For discretizing continuous time models with unequally-spaced date, the previous chapter provides a method, which imposes restrictions on the parameter matrix $A$ to be nonsingular. This, however, rules out applications to nonstationary systems such as unit root and cointegrated systems. This chapter presents an alternative method to deriving the exact discrete time representation for continuous time models with unequally-spaced flows and mixed data. In all cases the discrete time representations follow a VARMA$(1, 1)$ process with time-varying parameters and heteroskedasticity, despite that the underlying continuous time model has constant parameters and homoskedasticity. The time-varying parameters and the heteroskedastic variance arise due to the variations in the sampling intervals, whereas the moving average disturbances arise due to the flow nature of the observations. The exact discrete time representation for flow variables can be applied to nonstationary systems such as unit root and cointegrated systems since it imposed no restrictions on the matrix $A$; while the exact discrete time model for mixed samples requires the assumption that $A^{ss}$ is nonsingular. This restriction limits the potential applications to systems involving zero roots and cointegration between the stocks.

A Monte Carlo simulation study is conducted, aiming at examining estimates properties for the model, which correctly measures the unequal sampling intervals. The main procedure of the study is to simulate unequally-spaced data (monthly data) and then estimate the continuous time parameters using the exact discrete time model, which accounts for the unequal sampling intervals. Comparing to estimations results, based on the simulated data, from using the model, which treats sampling intervals as equal, the simulation results suggest that estimation bias is reduced when the (unequal) sampling intervals are measured correctly.

In the Monte Carlo study, we only simulate monthly data, which presents relatively small variation in sampling intervals. Though the simulation evidence indicates the favour of exact discrete time models accounting for the irregularity of sampling intervals, the estimation results are close when using different discrete time models. These relatively small estimation bias discrepancies may be explained by the small variations in the sampling intervals. With more irregularly spaced data, the advantage of accounting for the unequal sampling intervals could get bigger. Another potential extended work could be deriving the exact discrete time representation for mixed data, which does not impose restriction on matrix $A^{ss}$, such that the results could have broader applications of interest. This possibly request a different method which does not require inverting matrix $A^{ss}$.

# 6 REFERENCE

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# 7 APPENDIX

## 7.1 Proof of Theorem 2.1

Consider a system of stochastic differential equations where an intercept and a deterministic time trend are included:

$$dx(t) = [\mu + \gamma t + Ax(t)]dt + \zeta(dt), \quad t > 0, \tag{25}$$

where $x(t)$ is an $(n \times 1)$ vector of random processes, $\mu$ is an $(n \times 1)$vector of unknown constants, $\gamma t$ is an $(n \times 1)$ vector of deterministic time trend with $\gamma$ being the unknown slope and A is an $(n \times n)$ matrix of unknown coefficients. The disturbance vector, $\zeta(dt)$, is assumed to be a vector stochastic process which has the following properties:

**Assumption 2.1**.

$$E[\zeta(dt)] = 0$$

$$E[\zeta(dt)\zeta(dt)'] = \Sigma dt,$$

where $\Sigma$ is an unknown symmetric positive definite matrix and

$$E[\zeta_i(\Delta_1)\zeta_j(\Delta_2)'] = 0,$$

for $i, j = 1, 2, ..., n$; $i \neq j$; and $\Delta_1 \cap \Delta_2 = \emptyset$.

The system (25) is loosely described as a closed-form linear system of first order stochastic differential equations. We shall consider a system that only includes stock variables $x(t_i) = [x_1(t_i), x_2(t_i), ..., x_n(t_i)]'$, that are observed at each discrete point of time $t_i$, with $i = 1, 2, ..., T$. The sample interval is defined as $\delta_i = t_i - t_{i-1}$ for $i = 1, 2, ..., T$, which might not be equal to unity.

The system of stock variables, based on solution to system (25) [3] , can be written as

$$x(t_i) = \int_0^{t_i} e^{(t_i - r)A}\,\zeta(dr) + e^{t_i A}x(0) + \int_0^{t_i} e^{(t_i - r)A}[\mu + \gamma r]\,dr, \tag{26}$$

with boundary conditions $x(0) = \alpha$ for $t_0 = 0$ and $\alpha$ is any constant vector such that at time $t = 0$, the observation $x(0)$ is pre-determined.

Given that Assumption 2.1 is satisfied, the exact discrete time representation of system (26) is given by Theorem 2.1.

[3] further details can be found in Bergstrom (1983, 1984)

**Theorem 2.1**. Let $x(t)$ be generated by (25). Then, under Assumption 2.1, subject to the boundary condition, the discrete time data satisfy

$$x(t_i) = F_i x(t_{i-1}) + c_{0i} + c_{1i} t_i + \eta(t_i), \quad i = 1, ..., T. \tag{27}$$

$$E[\eta(t_i)] = 0,$$

$$E[\eta(t_i)\eta(t_i)'] \quad = \quad \Omega_i = \int_0^{\delta_i} [e^{rA} \Sigma e^{rA'}] dr,$$

$$E[\eta(t_i)\eta(t_j)'] = 0 \quad for \quad i \neq j,$$

where

$$c_{0i} = G_i \mu - H_i \gamma,$$

$$c_{1i} = G_i \gamma,$$

$$F_i = e^{\delta_i A}$$

$$G_i = \int_0^{\delta_i} e^{sA}\, ds,$$

$$H_i = \int_0^{\delta_i} e^{sA} s\, ds,$$

**Proof of Theorem 2.1**. The derivation of the exact discrete model of (26) is straightforward. By partitioning (26)

$$\begin{aligned}
x(t_i) \quad &= \quad \int_0^{t_{i-1}} e^{(t_i - r)A}\, \zeta(dr) + \int_{t_{i-1}}^{t_i} e^{(t_i - r)A}\, \zeta(dr) + e^{t_i A} x(0) \\
&+ \quad \int_0^{t_{i-1}} e^{(t_i - r)A} [\mu + \gamma r]\, dr + \int_{t_{i-1}}^{t_i} e^{(t_i - r)A} [\mu + \gamma r]\, dr \\
&= \quad e^{\delta_i A} \left\{ \int_0^{t_{i-1}} e^{(t_{i-1} - r)A}\, \zeta(dr) + e^{t_{i-1} A} x(0) + \int_0^{t_{i-1}} e^{(t_{i-1} - r)A} [\mu + \gamma r]\, dr \right\} \\
&+ \quad \int_{t_{i-1}}^{t_i} e^{(t_i - r)A} [\mu + \gamma r]\, dr + \int_{t_{i-1}}^{t_i} e^{(t_i - r)A}\, \zeta(dr) \\
&= \quad e^{\delta_i A} x(t_{i-1}) + \int_{t_{i-1}}^{t_i} e^{(t_i - r)A} [\mu + \gamma r]\, dr + \int_{t_{i-1}}^{t_i} e^{(t_i - r)A}\, \zeta(dr),
\end{aligned} \tag{28}$$

we obtain (27) with

$$\begin{aligned}
c_i &= \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}[\mu+\gamma r]\,dr, \\
&= \int_0^{\delta_i} e^{sA}\,ds\mu + \int_0^{\delta_i} e^{sA}(t_i-s)\,ds\gamma \\
&= \int_0^{\delta_i} e^{sA}\,ds\mu - \int_0^{\delta_i} e^{sA}s\,ds\gamma + \int_0^{\delta_i} e^{sA}\,ds\gamma t_i \\
&= c_{0i} + c_{1i}t_i
\end{aligned}$$

and

$$\eta(t_i) = \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,\zeta(dr).$$

The properties of the discrete time disturbance vector $\eta(t_i)$ are derived as follows:

The mean of vector $\eta(t_i)$ is obtained as

$$\begin{aligned}
E[\eta(t_i)] &= E\left[\int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,\zeta(dr)\right] \\
&= \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,E[\zeta(dr)] \\
&= 0.
\end{aligned}$$

The variance of $\eta(t_i)$ is obtained as

$$\begin{aligned}
E[\eta(t_i)\eta(t_i)'] &= E\left[\int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,\zeta(dr)\right]\left[\int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,\zeta(dr)\right]' \\
&= \int_{t_{i-1}}^{t_i} [e^{(t_i-r)A}\Sigma e^{(t_i-r)A'}]\,dr \\
&= \int_0^{\delta_i} [e^{rA}\Sigma e^{rA'}]dr.
\end{aligned}$$

The autocovariances of $\eta(t_i)$ is obtained as

$$\begin{aligned}
E[\eta(t_i)\eta(t_j)'] &= E\left[\int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,\zeta(dr)\right]\left[\int_{t_{j-1}}^{t_j} e^{(t_j-r)A}\,\zeta(dr)\right]' \\
&= 0,
\end{aligned}$$

for $i \neq j$. Since $i \neq j$ implies $\delta_i \neq \delta_j$, and hence $[t_{i-1}, t_i] \cap [t_{j-1}, t_j] = \emptyset$. **End of proof.**

### 7.2 Cholesky factorization of the covariance matrix $\Omega$

Let $M$ be the real $nT \times nT$ lower triangular matrix with positive elements along the diagonal:

$$M = \begin{bmatrix} M_{11} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ M_{21} & M_{22} & 0 & \cdots & \cdots & 0 & 0 \\ 0 & M_{32} & M_{33} & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & M_{T-1,T-2} & M_{T-1,T-1} & 0 \\ 0 & 0 & \cdots & \cdots & 0 & M_{T,T-1} & M_{T,T} \end{bmatrix}.$$

The sub-matrices, $M_{11}, \cdots, M_{t,t-1}, M_{tt}(t = 2, \cdots, T)$ can be computed as

$$\begin{aligned} M_{11}M'_{11} &= \Omega_0, \\ M_{21} &= \Omega_1(M'_{11})^{-1}, \\ M_{22}M'_{22} &= \Omega_0 - M_{21}M'_{21}, \\ &\vdots \\ M_{t,t-1} &= \Omega_1(M'_{t-1,t-1})^{-1}, \\ M_{tt}M'_{tt} &= \Omega_0 - M_{t,t-1}M'_{t,t-1}, t = 2, \cdots, T. \end{aligned}$$

To compute $M$, we need to compute (elements of ) $\Omega$; then we need to compute $\xi$. It is necessary to define a normally distributed $nT \times 1$ vector satisfying $M\epsilon = \xi$: Define an $nT \times 1$ vector $\epsilon = [\epsilon'_1, \cdots, \epsilon'_T]'$ such that $M\epsilon = \xi$, where $E[\epsilon] = 0, E[\epsilon\epsilon'] = I_{nT}$ and $E[\epsilon_t] = 0, E[\epsilon_t\epsilon'_t] = I_n, E[\epsilon_t\epsilon'_s] = 0$ for $s \neq t$ and $s, t = 1, \cdots, T$. Therefore, $\xi'\Omega^{-1}\xi = \xi'(M')^{-1}M^{-1}\xi = \epsilon'\epsilon$. Then $\xi$ is computed as $\xi = M\epsilon$, whose procedure is given in section 4.

### 7.3 Computing elements of $\Omega$

Elements of the matrix $\Omega$ include

$$\begin{aligned}\Omega_{01} &= \int_0^{\delta_1} G(s)\Sigma G(s)'ds \\ &= \int_0^{\delta_1}\int_0^s\int_0^s e^{Ar}\Sigma e^{A'w}dwdrds \\ &= \Psi(\delta_1),\end{aligned}$$

$$\begin{aligned}\Omega_{1,i} &= \int_0^{\delta_i} G(s)\Sigma\Gamma(s)'ds \\ &= \left(\int_0^{\delta_i}\int_0^s e^{Ar}\Sigma e^{A's}drds\right)G'_{i+1} - \left(\int_0^{\delta_i}\int_0^s\int_0^s e^{Ar}\Sigma e^{A'w}dwdrds\right)\Phi'_{i+1}, \\ &= \Lambda(\delta_i)G'_{i+1} - \Psi(\delta_i)\Phi'_{i+1},\end{aligned}$$

where $\Lambda(\delta_i) = \int_0^{\delta_i}\int_0^s e^{Ar}\Sigma e^{A's}drds = \int_0^{\delta_i} G(s)\Sigma F(s)'ds$, and $\Psi(\delta_i) = \int_0^{\delta_i}\int_0^s\int_0^s e^{Ar}\Sigma e^{A'w}dwdrds = \int_0^{\delta_i} G(s)\Sigma G(s)'ds, i = 2, \cdots, T$ in the following,

$$\begin{aligned}\Omega_{-1,i} &= \int_0^{\delta_{i-1}} \Gamma_i(s)\Sigma G(s)'ds \\ &= G_i\Lambda(\delta_{i-1})' - \Phi_i\Psi(\delta_{i-1})',\end{aligned}$$

$$\begin{aligned}\Omega_{0,i} &= \int_0^{\delta_i} G(s)\Sigma G(s)'ds + \int_0^{\delta_{i-1}} \Gamma_i(s)\Sigma\Gamma_i(s)'ds \\ &= \Psi(\delta_i) + G_iL(\delta_{i-1})G'_i - G_i\Lambda(\delta_{i-1})'\Phi'_i - \Phi_i\Lambda(\delta_{i-1})G'_i + \Phi_i\Psi(\delta_{i-1})\Phi'_i,\end{aligned}$$

where $Ł(\delta_i) = \int_0^{\delta_i} e^{As}\Sigma e^{A's}ds = \int_0^{\delta_i} F(s)\Sigma F(s)'ds$.

In order to compute elements of $\Omega$, we need to compute following matrix exponential and its integrals:

$$F_i = e^{\delta_i A}, G_i = \int_0^{\delta_i} e^{As}ds, L(\delta_i) = \int_0^{\delta_i} e^{As}\Sigma e^{A's}dsds,$$
$$\Lambda(\delta_i) = \int_0^{\delta_i}\int_0^s e^{Ar}\Sigma e^{A's}drds, \Psi(\delta_i) = \int_0^{\delta_i}\int_0^s\int_0^s e^{Ar}\Sigma e^{A'w}dwdrds.$$

Since the matrix $A$ is singular, we cannot directly compute $\int_0^{\delta_i} e^{As}ds = A^{-1}(e^{\delta_i A} - I)$. The matrix exponential, $e^{A\delta_i}$, and the integrals of the matrix exponential can be obtained from the computation of a $4n \times 4n$ matrix exponential (see Van Loan, 1978; and Thornton and Chambers, 2016). Let $C$ be the $4n \times 4n$ upper triangular matrix, defined by

$$C = \begin{bmatrix} -A & I & 0 & 0 \\ 0 & -A & \Sigma & 0 \\ 0 & 0 & A' & I \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then for $\delta_i \geqslant 0$ for all $i$,

$$e^{c\delta_i} = exp\left\{ \delta_i \begin{bmatrix} -A & I & 0 & 0 \\ 0 & -A & \Sigma & 0 \\ 0 & 0 & A' & I \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} = \begin{bmatrix} F_1(\delta_i) & G_1(\delta_i) & H_1(\delta_i) & K_1(\delta_i) \\ 0 & F_2(\delta_i) & G_2(\delta_i) & H_2(\delta_i) \\ 0 & 0 & F_3(\delta_i) & G_3(\delta_i) \\ 0 & 0 & 0 & F_3(\delta_i) \end{bmatrix},$$

where

$$F_3(\delta_i) = e^{A'\delta_i},$$

$$\begin{aligned} G_2(\delta_i) &= \int_0^{\delta_i} e^{-A(\delta_i - s)} \Sigma e^{A's} ds \\ &= e^{-A\delta_i} \int_0^{\delta_i} e^{As} \Sigma e^{A's} ds, \end{aligned}$$

$$G_3(\delta_i) = \int_0^{\delta_i} e^{A(\delta_i - s)} ds,$$

$$H_2(\delta_i) = e^{-A\delta_i} \int_0^{\delta_i} \int_0^{\delta_s} e^{As} \Sigma e^{A'r} dr ds,$$

Therefore,

$$F_i = F_3(\delta_i)',$$

$$G_i = G_3(\delta_i),$$

$$L(\delta_i) = F_3(\delta_i)' G_2(\delta_i),$$

$$\Lambda(\delta_i) = F_3(\delta_i)' H_2(\delta_i),$$

$$\Psi(\delta_i) = F_3(\delta_i)' K_1(\delta_i) + K_1(\delta_i)' F_3(\delta_i).$$

### 7.4 Derivation of "equally-spaced" model

In the case of model with equally spaced flows, observations, $x_t$, are made over equally spaced discrete integrals, $(t-1,t)$, such that $x_t = \int_{t-1}^{t} x(r)dr, t = 1, \cdots, T$.

Let $x(t)$ be an $n \times 1$ stochastic process generated by

$$dx(t) = Ax(t)dt + \zeta(dt), \quad t > 0,$$

where $A = \begin{bmatrix} \alpha_1 & -\alpha_1\beta \\ \alpha_2 & -\alpha_2\beta \end{bmatrix}$.

If $x(t)$ is a stock variable, then the discrete time form of (1) obtained as

$$x(t) = Fx(t-1) + \eta(t), t = 1, \cdots, T,$$

where $F = e^{A\delta} = e^{A}$, given that $\delta = 1$, $\eta(t) = \int_{t-1}^{t} e^{A(t-r)}\zeta(dr)$.

If the observations are flow variables then

$$x_t = \int_{t-1}^{t} x(r)dr = \int_{0}^{1} x(t-r)dr = \int_{0}^{1} x(t-1+r)dr.$$

From the discrete time model for stock variables we obtain

$$x(t-1+s) = e^{As}x(t-r) + \int_{t-1}^{t-1+s} \zeta(dr).$$

Integrating the above equation over the interval $s \in (0,h]$ obtains

$$\int_{0}^{1} x(t-1+s)ds = (\int_{0}^{1} e^{As})x(t-1) + \int_{0}^{1}\int_{t-1}^{t-1+s} e^{A(t-1+s-r)}\zeta(dr)ds,$$

which can be represented as

$$x_t = Gx(t-1) + e_t,$$

where $G = \int_0^1 e^{As}ds, e_t = \int_0^1 \int_{t-1}^{t-1+s} e^{A(t-1+s-r)}\zeta(dr)ds = \int_{t-1}^{t} G(t-r)\zeta(dr), t = 1, \cdots, T$.

Re-arranging the above equation yields

$$x(t-1) = G^{-1}(x_t - e_t).$$

Lagging the discrete time model for stocks for one period and substituting out $x(t-1)$ using the above equation obtains

$$G^{-1}(x_t - e_t) = FG^{-1}(x_{t-1} - e_{t-1}) + \eta(t-1).$$

Re-arranging the above equation obtains the reduced-form discrete time model

$$x_1 = \Phi x(0) + \epsilon_1,$$

$$x_t = \Phi x_{t-1} + \epsilon_t, \quad t = 2, \ldots, T,$$

where $\epsilon_1 = e_1 = \int_0^1 G(1-r)\zeta(dr)$ with $t(0) = 0$ and $t_1 = \delta = 1, \epsilon_t = \int_{t-1}^t G(t-r)\zeta(dr) + \int_{t-2}^{t-1} \Gamma(t-1-r)\zeta(dr), t = 2, \ldots, T$.

Properties of the disturbances are given by

$$\Omega_0 = E[\epsilon_t \epsilon_t'] = \begin{cases} \int_0^1 G(s)\Sigma G(s)' ds, & t = 1, \\ \int_0^1 G(s)\Sigma G(s)' ds + \int_0^1 \Gamma(s)\Sigma\Gamma(s)' ds, & t = 2, \ldots, T, \end{cases}$$

$$\Omega_{-1} = E[\epsilon_t \epsilon_{t-1}'] = \int_0^1 \Gamma(s)\Sigma G(s)' ds, \quad t = 2, \ldots, T,$$

$$\Omega_1 = E[\epsilon_t \epsilon_{t+1}'] = \int_0^1 G(s)\Sigma\Gamma(s)' ds, \quad t = 1, \ldots, T-1,$$

Furthermore, let $\Omega_{00}$ denote the variance when $t = 1$ and the we have

$$\Omega_{00} = \int_0^1 G(s)\Sigma G(s)' ds,$$

and the covariance matrix is

$$\begin{aligned} \Omega &= E\left[\epsilon\epsilon'\right] \\ &= \begin{bmatrix} \Omega_{00} & \Omega_1 & 0 & 0 & \cdots & \cdots & 0 \\ \Omega_{-1} & \Omega_0 & \Omega_1 & 0 & \cdots & \cdots & 0 \\ 0 & \Omega_{-1} & \Omega_0 & \Omega_1 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & \Omega_{-1} & \Omega_0 & \Omega_1 \\ 0 & 0 & \cdots & \cdots & 0 & \Omega_{-1} & \Omega_0 \end{bmatrix}. \end{aligned}$$

The simulation procedure is the same as in the "unequally-spaced" model.